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Edge waves on a gently sloping beach: uniform asymptotics

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Edge waves on a beach of gentle slope $\epsilon \ll 1$ are considered. For constant slope, Ursell (1952) has obtained a complete set of trapped modes and shown that there exists only a finite number n of such modes, $(2n+1)\beta < \frac{1}{2}\pi$, $\beta = \tan^{-1}\epsilon$. For non-uniform slope the formulae for the trapped-mode frequencies were heuristically derived by Shen, Meyer & Keller (1968). For small $n \sim O(1)$ Miles (1989) has obtained formulae which coincide with Shen *et al.*'s (1968) with accuracy to $O(\epsilon)$ and differ from them by $O(\epsilon^2)$. However, Miles' formulae fail at $n \sim 1/\epsilon$. In this paper it is proved that Shen *et al.*'s (1968) formulae are valid for all n (including $n \sim 1/\epsilon$) with accuracy to $O(\epsilon)$ and corrections of any order in ϵ are given. Uniform asymptotic expansions are obtained for the corresponding eigenfunctions. These expansions give Miles' (1989) result for small n. The formulae for the frequencies and the eigenfunctions have the same structure for both the full dispersion system and the shallow-water equation. For small n the frequencies for both models coincide with accuracy to $O(\epsilon^2)$, but for $n \sim 1/\epsilon$ they differ by $O(1)$. In the last section the effect of rotation following Evans (1989) is taken into account. All the asymptotics have formal character, i.e. they satisfy the corresponding equations with accuracy to $O(\epsilon^N)$, N being arbitrarily large. The rigorous justification of these asymptotics is under way.

1. Formulation of the problem and main results

Consider a basin of depth $z = -H(y)$, $0 \leqslant y < \infty$. The shore corresponds to $z = 0$, $y = 0$, $H(0) = 0$. We assume that the slope is small, $H'(y) \ll 1$, and the function H has the form $kH(y) = h(\eta)$, where $\eta = \epsilon k y$, $\epsilon = H'(0) \ll 1$, k is the longshore wavenumber, the function $h(\eta)$ is smooth and analytic in a vicinity of the point $\eta = 0$ and $C_1\eta \geqslant h(\eta) \geqslant C_2$ as $\eta \to \infty$ for some positive constants C_1, C_2. Thus we include the cases when the depth is constant or increases at infinity. Setting $\Phi = A\cos(\omega t - kx)\,\phi(y,z)$ (Φ is the velocity potential and x is the longshore coordinate) and introducing the coordinate $\zeta = kz$, we get

$$\phi_{\zeta\zeta} + \epsilon^2\phi_{\eta\eta} - \phi = 0, \quad -h(\eta) < \zeta < 0, \tag{1.1}$$

$$\phi_\zeta = \lambda\phi, \quad \zeta = 0, \tag{1.2}$$

$$\phi_\zeta + \epsilon^2 h'\phi_\eta = 0, \quad \zeta = -h(\eta), \tag{1.3}$$

where $\lambda = \omega^2/gk$.

Miles (1985) reduced the system (1.1)–(1.3) to one equation by using the following argument. Representing the solution of this system in the form

$$\phi = \int \exp(\mathrm{i}p\eta/\epsilon)[\cosh(\kappa\zeta) + \lambda\kappa^{-1}\sinh(\kappa\zeta)]f(p)\,\mathrm{d}p, \tag{1.4}$$

where $\kappa = (p^2+1)^{\frac{1}{2}}$, $f(p)$ is a new unknown function and the integration is carried along some contour in the complex plane p, we see that (1.1) and (1.2) are satisfied exactly and (1.3) gives an integral equation

$$\int \exp(\mathrm{i}p\eta/\epsilon)\, L(\eta,p) f(p)\,\mathrm{d}p = 0 \tag{1.5}$$

for the function $f(p)$. Here

$$L = L_0 + \mathrm{i}\epsilon L_1, \quad L_0(\eta,p,\lambda) = \lambda\cosh(\kappa h) - \kappa\sinh(\kappa h),$$
$$L_1(\eta,p,\lambda) = ph'[\cosh(\kappa h) - \lambda\kappa^{-1}\sinh(\kappa h)].$$

Equation (1.5) differs from (2.5) in Miles (1989) in notation only. The function $Z(\eta) = \phi(\eta, 0)$ which gives the free-surface displacement within a factor is given by

$$Z(\eta) = \int \exp(\mathrm{i}p\eta/\epsilon) f(p)\,\mathrm{d}p, \tag{1.6}$$

and the edge-wave boundary conditions

$$|Z(0)| < \infty, \quad Z(\eta) \to 0 \quad (\eta \to \infty) \tag{1.7}$$

provide the appropriate conditions for the function $f(p)$.

For the description of edge waves, the shallow-water equation

$$\epsilon^2(hZ')' - hZ + \lambda Z = 0 \tag{1.8}$$

is also used. This equation is easily rewritten in the form (1.5) in which $L_0^{\mathrm{s}} = \lambda_{\mathrm{s}} - \kappa^2 h$ and $L_1^{\mathrm{s}} = h'p$ are used instead of L_0 and L_1. Obviously, $L_0 = L_0^{\mathrm{s}} + O(h^2), L_1 = L_1^{\mathrm{s}} + O(h)$.

Miles (1989) has shown that solutions of the problem (1.5), (1.7) exist only if λ satisfies

$$\lambda = \epsilon(2n+1) + \epsilon^2 h''(0)(n^2+n+\tfrac{1}{2}) + O(\epsilon^3), \quad n = 0, 1, \ldots. \tag{1.9}$$

On the other hand, in the case $h = \eta$, Ursell's (1952) result gives

$$\lambda = \sin[(2n+1)\beta], (2n+1)\beta < \tfrac{1}{2}\pi, \quad \beta = \tan^{-1}\epsilon. \tag{1.10}$$

Expanding in ϵ for $n \sim O(1)$ we get (1.9), while for $n \sim 1/\epsilon$ (i.e. $(2n+1)\epsilon \sim O(1)$)

$$\lambda = \sin(2n+1)\epsilon - \tfrac{1}{3}\epsilon^2[(2n+1)\epsilon]\cos(2n+1)\epsilon + O(\epsilon^4). \tag{1.11}$$

Obviously, for such n (1.9) fails.

A similar argument is valid for the shallow-water equation. For $h = \eta$ (1.9) gives the exact result, (1.8) being the Laguerre equation, while for Ball's (1967) profile $h(\eta) = \alpha(1 - \exp(-\eta/\alpha))$, $\alpha = kH(\infty)$, equation (1.9) is valid only for small n. Indeed, in this case the exact formula has the form

$$\lambda = \epsilon[(2n+1)(1+\epsilon^2/4\alpha^2)^{\frac{1}{2}} - \epsilon(n^2+n+\tfrac{1}{2})/\alpha],$$

and for $(2n+1)\epsilon \sim O(1)$ the expansion in ϵ differs from (1.9) by $\epsilon^2[(2n+1)\epsilon]/8\alpha^2$ in the $O(\epsilon^2)$ term.

Another approach to the problem (1.5), (1.7) was elaborated by Shen, Meyer & Keller (1968). Using the geometrical optics approximation, they proposed the formula

$$\frac{1}{\pi\epsilon}\int_0^{q_0} p(q,\lambda)\,\mathrm{d}q = n + \tfrac{1}{2}, \quad n = 0, 1, \ldots,$$

where $p(q, \lambda)$ is the positive solution of the equation

$$\kappa(p) \tanh(\kappa(p)h(q)) = \lambda, \tag{1.12}$$

q_0 is the turning point, $\tanh h(q_0) = \lambda$. Integrating by parts and changing the variables, $p(q, \lambda) \mapsto p$, we get

$$\frac{1}{\pi\epsilon}\int_{-\infty}^{+\infty} q(p, \lambda)\, \mathrm{d}p = 2n+1, \tag{1.13}$$

where $q(p, \lambda)$ is the solution of the same equation (1.12). Elementary calculations show that the left-hand side of (1.13) is a monotonically increasing function of λ for monotonic $h(\eta)$. The solution of (1.12) is a smooth integrable function only for sufficiently small $\lambda < \tanh h(\infty)$. Therefore, (1.13) is solvable only for a finite number of n, and its solution $\lambda = \lambda_0(\sigma), \sigma = (2n+1)\epsilon$, depends regularly on σ for $\lambda_0(\sigma) < \tanh h(\infty)$.

According to Miles (1989), for small n (1.13) differs from (1.9) by $\frac{1}{4}\epsilon^2 h''(0)$. Nevertheless, for $h = \eta$ (1.13) gives $\lambda = \sin(2n+1)\epsilon$, which coincides with (1.10) with accuracy to $O(\epsilon^2)$ even for large n.

Therefore two natural questions arise: (i) In what sense is (1.13) valid? (ii) What is the relation between (1.9) and (1.13)?

Here are some answers. The eigenfrequencies in (1.5) have the form

$$\lambda = \lambda_0 + \epsilon^2\lambda_1 + \epsilon^3\lambda_2 + \dots, \tag{1.14}$$

where λ_j depend on σ, λ_0 is the solution of (1.12), (1.13). The correction λ_1 is given by the formula

$$\lambda_1 \int_{-\infty}^{+\infty} a_0^2\, \mathrm{d}p = \int_{-\infty}^{+\infty} a_0 F/b\, \mathrm{d}p, \tag{1.15}$$

where $$a_0(p) = b(p)/\kappa(h'(p))^{\frac{1}{2}}, b(p) = \cosh[\kappa h(q)], q = q(p, \lambda_0),$$

$$F(p) = a_0[L_{1\eta p} + \tfrac{1}{2}q_p(L_{1\eta\eta} + L_{0\eta\eta\eta p}) + \tfrac{1}{2}L_{0\eta\eta pp} + \tfrac{1}{6}q_{pp}L_{0\eta\eta\eta}$$
$$+ \tfrac{1}{8}q_p^2 L_{0\eta\eta\eta\eta}] + a_0'(L_{1\eta} + L_{0\eta\eta p} + \tfrac{1}{2}q_p L_{0\eta\eta\eta}) + \tfrac{1}{2}a_0'' L_{0\eta\eta}.$$

Here the arguments $(q(p, \lambda_0), p, \lambda_0)$ of the derivatives of the functions $L_{0,1}$ are omitted. The formulae for $\lambda_j, j \geqslant 2$ are given in §2.

Thus the formula (1.13) is valid for all n with accuracy to $O(\epsilon)$. For small n (1.15) gives $\lambda_1 = \frac{1}{4}h''(0) + O(\epsilon)$ and (1.14) transforms into (1.9). For $h = \eta$ (1.15), after rather tedious calculations, gives the second term in (1.11).

An analogous result is valid for the shallow-water equation. Namely, the eigenfrequency λ is given by (1.14) where λ_0 is defined by (1.13), the function q being replaced by the solution $q_s(p, \lambda_0)$ of the equation $\kappa^2 h(q_s) = \lambda_0$, and λ_1 is given by (1.15) where $b = 1$, $q = q_s$, $L_{0,1} = L_{0,1}^s$. For small n this formula again gives (1.9). It must be noted that in this case (1.13) is solvable either for all n or for a finite number of n only, depending on whether the function h increases or has a finite limit at infinity.

These results are obtained through explicit evaluation of the eigenfunctions. Shen *et al.* (1968) noted that the eigenfunctions may be obtained by means of the matching method. Subsequently, Shen & Keller (1975) constructed a uniform asymptotic approximation based on special functions using the ideas of Langer (see Olver 1974) and Ludvig (1966). However, they did not carry out detailed calculations for the case under consideration here. We note that perhaps it is more appropriate to use a model equation of the form

$$V''(\zeta)+\zeta^{-1}V'(\zeta)+\epsilon^{-2}\zeta^{-1}(\zeta_1-\zeta)V(\zeta)=0, \tag{1.16}$$

$\zeta=\zeta_1$ corresponding to the caustic and ζ being the phase in Shen & Keller's notation, instead of their equation (6.8), and the equation

$$V''(\epsilon^{-2}\zeta)+\epsilon^2\zeta^{-1}V'(\epsilon^{-2}\zeta)+\epsilon^2\zeta^{-1}V(\epsilon^{-2}\zeta)=0 \tag{1.17}$$

with solution $V(\tau)=J_0(2\tau^{\frac{1}{2}})$ instead of the Bessel equation in the case of a single shoreline, because the use of (1.16), (1.17) does not involve the square-root singularities of the phase function ζ at the shoreline, contrary to (6.8) of Shen & Keller (1975) (see formula (C.11) of their paper). This weak singularity of the phase does not affect the validity of the leading term of the asymptotics but can hinder the construction of corrections. It turns out that the uniform (simultaneously in η and n) asymptotics of the eigenfunctions can be obtained in the form (1.6), where

$$f(p)=\frac{1}{(2\pi\epsilon)^{\frac{1}{2}}}a(p,\epsilon)\exp(-\mathrm{i}S(p)/\epsilon),\quad a=a_0(p)+\epsilon a_1(p)+\ldots, \tag{1.18}$$

so that $Z(\eta)$ is an inverse Fourier transform of a rapidly oscillating exponential (cf. Maslov 1972). The functions S, a_j in (1.18) depend parametrically on σ. The function $Z(\eta)$ satisfies the boundary conditions (1.7) if (1.13), (1.15) are fulfilled. At small n we get modulo a factor

$$Z(\eta)\sim \mathrm{L}_n(2\eta/\epsilon)\exp(-\eta/\epsilon), \tag{1.19}$$

where L_n are Laguerre polynomials (see §3). Equation (1.19) coincides with Miles' result

$$Z(\eta)\sim \mathrm{L}_n(2h/\epsilon)\exp(-h/\epsilon), \tag{1.20}$$

because $\eta^N\exp(-\eta/\epsilon)=O(\epsilon^N)$ for $\eta\geqslant 0$ and all N, so that the function h in (1.20) may be expanded in powers of η and only the leading terms are substantial. We note also that asymptotic form of Ursell's solution for small n and β coincides with (1.19). This fact is easily obtained from the integral representation of the eigenfunctions (see Whitham 1975). Thus in this case the geometrical optics approximation (modified in the sense of Maslov 1972) is valid for modes of small number.

For large n the structure of the eigenfunctions is somewhat more complicated. At the internal points of the interval $(0,q_0)$, where q_0 is the turning point, $\tanh h(q_0)=\lambda_0$, they are expressed through the standard WKB formulae, in the vicinity of the turning point – through the Airey function and in the vicinity of the point $\eta=0$ – in terms of the Bessel function J_0 (see §3).

All these results can be obtained by means of (1.16), its solution having the form

$$V(\zeta)=\mathrm{e}^{-\zeta/\epsilon}M\left(\tfrac{1}{2}-\frac{\zeta_1}{2\epsilon},1,\frac{2\zeta}{\epsilon}\right), \tag{1.21}$$

where $M(a,b,z)$ is the confluent hypergeometric function. However, the construction of corrections becomes much more tedious and the geometric interpretation of §2 is not so clear when this method is used. The results of §3 can be obtained by means of the asymptotic expression of (1.18) through (1.21) according to the generalized stationary phase method (see Olver 1974), the asymptotic forms of the confluent hypergeometric function providing the non-uniform formulae mentioned above. We do not give here the corresponding calculations because they are rather lengthy and eventually lead to the same results.

2. Construction of the asymptotic eigenfunctions

Supposing that the integration in (1.5) is carried along a closed contour C and integrating by parts, we get

$$I = \int_C \exp\,[\mathrm{i}(p\eta - S)/\epsilon]\, L(\eta, p)\, a(p)\, \mathrm{d}p$$

$$= \int_C \exp\,[\mathrm{i}(p\eta - S)/\epsilon]\, L(S_p, p)\, a(p)\, \mathrm{d}p - \frac{\epsilon}{\mathrm{i}} \int_C \exp\,[\mathrm{i}(p\eta - S)/\epsilon]\, \mathscr{L}\,[La](\eta, p)\, \mathrm{d}p, \quad (2.1)$$

where the operator $\mathscr{L}$ is defined by the equality

$$\mathscr{L}[f(\eta, p)](\eta, p) = \frac{\partial}{\partial p} \frac{f(\eta, p) - f(S_p, p)}{\eta - S_p}.$$

Carrying out this procedure with the second integral in (2.1) and reiterating, we get

$$I = \int_C \exp\,[\mathrm{i}(p\eta - S)/\epsilon]\{L(S_p, p)\, a + \mathrm{i}\epsilon \mathscr{L}[La](S_p, p)$$
$$- \epsilon^2 \mathscr{L}^2[La](S_p, p) + \ldots + (\mathrm{i}\epsilon)^n \mathscr{L}^n[La](S_p, p) + \ldots\}\, \mathrm{d}p. \quad (2.2)$$

Substituting the expansions $a = a_0 + \epsilon a_1 + \ldots, \lambda = \lambda_0 + \epsilon^2 \lambda_1 + \ldots$ into (2.2) and equating to zero the coefficients at $\epsilon^0, \epsilon^1, \ldots$ in the integrand, we get

$$\sum_{j=0}^{m+1} \sum_{l=0}^{j} \mathrm{i}^{m-j+l+1}\, \mathscr{L}^{m-j+1}\,[L_l\, a_{j-l}](S_p, p) = 0, \quad m = -1, 0, 1, 2, \ldots, \quad (2.3)$$

where

$$L_2 = -\lambda_1 \cosh \kappa h(\eta), \quad L_l = \mathrm{i}^{-1}(\lambda_{l-1} \cosh \kappa h(\eta) - \mathrm{i}\lambda_{l-2}\, p h' \kappa^{-1} \sinh \kappa h(\eta)), \quad l \geqslant 3.$$

At $m = -1$ (2.3) gives (1.13) where $\lambda = \lambda_0$, $q = S_p$. This equation is equivalent to

$$\tfrac{1}{2}p^2 + U(q, \lambda_0) = -\tfrac{1}{2}, \quad (2.4)$$

where $U(q, \lambda) = -u^2(\lambda h(q))/2h^2(q)$, $u(z)$ is the inverse function to $z \tanh z$. The function $U \sim 1/q$ as $q \to 0$ and its graph for monotonic $h(\eta)$ is shown on figure 1. From the point of view of classical mechanics, (2.4) describes a particle of mass unity and momentum p which moves in a field with potential U at the energy level $E = -\frac{1}{2}$. The motion of this particle is finite when $U_0 = \lim_{q \to \infty} U(q, \lambda_0) > -\frac{1}{2}$, that is, $\lambda_0 < \tanh h(\infty)$. These motions correspond to the trapped wave modes. The phase trajectory of the particle is given by the equality $q = q(p, \lambda_0)$. The function q is smooth, if, say, $h'(q) > 0$ for $0 \leqslant q \leqslant q_0$ and $U_0 > -\frac{1}{2}$. A typical trajectory of this kind is shown on figure 2 by a continuous line. If $\lambda_0 \geqslant \tanh h(\infty)$, the phase trajectory splits into two separate parts, shown on figure 2 by dotted lines. This case corresponds to waves that belong to the continuous spectrum and do not vanish at infinity.

The equality $q = S_p$ gives

$$S = \int_0^p q(p, \lambda_0)\, \mathrm{d}p. \quad (2.5)$$

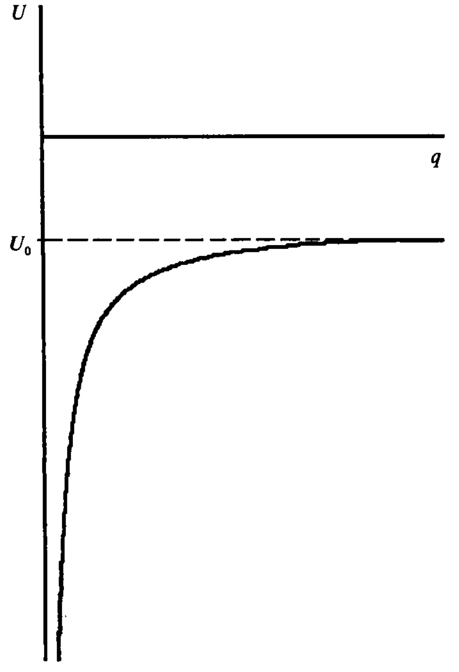

FIGURE 1. Effective potential.

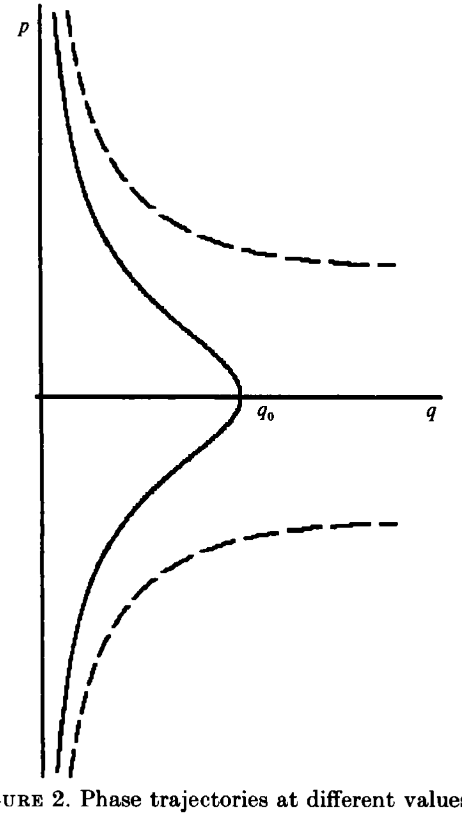

FIGURE 2. Phase trajectories at different values of λ.

Consider (2.3) for $m = 0$:

$$\mathscr{L}[L_0 a_0](S_p, p) + L_1 a_0 = 0. \tag{2.6}$$

Hadamard's lemma

$$f(y) - f(x) = (y-x)\int_0^1 f'(\theta y + (1-\theta)x)\,\mathrm{d}\theta$$

gives
$$\mathscr{L}[L_0 a_0](S_p, p) = L_{0\eta p} a_0 + L_{0\eta} a_{0p} + \tfrac{1}{2} L_{0\eta\eta} S_{pp} a_0$$

and (2.6) transforms into

$$L_{0\eta} a_{0p} + L_{0\eta p} a_0 + \tfrac{1}{2} L_{0\eta\eta} S_{pp} a_0 + L_1 a_0 = 0. \tag{2.7}$$

Using the identities

$$L_1(q,p) = \frac{M_{\eta p}(q,p)\cosh \kappa h(q)}{2(1-\kappa h(q)\tanh \kappa h(q))},$$

$$\tfrac{1}{2} M_{\eta p} + M_{\eta} p \kappa^{-1} h \tanh \kappa h - \frac{M_{\eta p}}{2(1-\kappa h \tanh \kappa h)} = 0,$$

where $M(\eta, p) = \kappa \tanh \kappa h(\eta)$, we get from (2.7)

$$\frac{\mathrm{d}}{\mathrm{d}p} a_0 M_{\eta}^{\frac{1}{2}}(q(p,\lambda_0), p) = 0. \tag{2.8}$$

Thus $a_0 = \text{const} \times M_{\eta}^{-\frac{1}{2}}$. Choosing const $= 1$, we get the main term of the eigenfunction

$$Z_0 \sim \frac{1}{(2\pi\epsilon)^{\frac{1}{2}}} \int_C \frac{b(q)}{\kappa (h'(q))^{\frac{1}{2}}} \exp\left[\frac{\mathrm{i}}{\epsilon}(p\eta - S)\right] \mathrm{d}p. \tag{2.9}$$

Consider now the analytic properties of the functions a_0, S. The function $q(p,\lambda)$ is analytic in p for sufficiently large $|p|$, say, $|p| > R$, owing to the analyticity of $h(\eta)$ for small η, $q \sim O(|p|^{-2})$, $|p| \to \infty$. Therefore, S is analytic for $|p| > R$ on the complex plane p with a slit along the imaginary axis. Choosing the branch of the function $(p^2+1)^{\frac{1}{2}}$ that coincides with arithmetic square root for real p, we see that the integrand in (2.9) is analytic on the plane with the slit for $|p| > R$. Now we choose the contour C in the following way: C consists of the part of the real axis for $|p| < R$ and the arc of a circle of radius R for $\operatorname{Im} p > 0$. On the left and right edges of the slit we have

$$S(\mathrm{i}R + 0) = S(\infty) - \int_{\mathrm{i}R+0}^{\infty} q\,\mathrm{d}p, \tag{2.10}$$

$$S(\mathrm{i}R - 0) = S(-\infty) - \int_{\mathrm{i}R-0}^{-\infty} q\,\mathrm{d}p. \tag{2.11}$$

Owing to the analyticity of q, the second summands in (2.10) and (2.11) are equal to each other. The full variation of the argument of the function κ along C equals π. Thus the full variation of the argument of the integrand in (2.9) equals $2S(\infty)/\epsilon - \pi$. We see now that if (1.13) is fulfilled, the integrand is analytic in the full neighbourhood of the infinity without the slit and (2.9) defines a function analytic in η, otherwise the integral in (2.9) taken along the real axis has a logarithmic singularity at the origin. Thus the first condition in (1.7) is fulfilled only if (1.13) holds. The condition at infinity is fulfilled because, for large η, $|\eta - S_p| \geqslant \text{const}\,\eta$ and it is possible to integrate (2.9) by parts to obtain $Z_0 = O(\epsilon^N \eta^{-N})$ for all N and $\eta \to \infty$.

Consider now (2.3) for $m \geqslant 1$. By the same argument as above, we get

$$\frac{\mathrm{d}}{\mathrm{d}p} a_m M_\eta^{\frac{1}{2}} = F_m, \tag{2.12}$$

where
$$F_m = -\mathrm{i}(\mathrm{i}^{m+1} L_{m+1} a_0 + G_m)/b(p) M_\eta^{\frac{1}{2}}(q,p),$$

$$G_m = \sum_{l=2}^{m} \mathrm{i}^l L_l a_{m-l+1} + \sum_{l=1}^{m} \mathrm{i}^{l+1} \mathscr{L}[L_l a_{m-l}](S_p, p)$$
$$+ \sum_{j=0}^{m-1} \sum_{l=0}^{j} \mathrm{i}^{m-j+l+1} \mathscr{L}^{m-j+1} [L_l a_{j-l}](S_p, p).$$

The solution of (2.12) is

$$a_m = M_\eta^{-\frac{1}{2}} \int_0^p F_m(p)\,\mathrm{d}p. \tag{2.13}$$

The function $a_m \exp[\mathrm{i}(p\eta - S)/\epsilon]$ is analytic at infinity only if

$$\int_0^\infty F_m\,\mathrm{d}p = \int_0^{-\infty} F_m\,\mathrm{d}p. \tag{2.14}$$

The value λ_m enters F_m only through the term containing L_{m+1} and (2.14) gives

$$\lambda_m \int_{-\infty}^{\infty} a_0^2\,\mathrm{d}p = -\int_{-\infty}^{\infty} G_m a_0/b\,\mathrm{d}p. \tag{2.15}$$

By induction it is easy to see that G_m is an even function for odd m and odd function for even m. Therefore $\lambda_{2l} = 0$, $l = 1, 2, \dots,$ and (2.15) defines λ_m for odd m. Rather tedious calculations show that (2.15) for $m = 1$ gives (1.15). Thus the full asymptotic expansion of $Z(\eta)$ is obtained through (1.6), (1.18).

Construction of the eigenfunctions for the shallow-water equation (1.8) is carried out according to the same formulae, the functions L_l being replaced by $L_l^{\mathrm{s}} = \mathrm{i}^{-1}\lambda_{l-1}$, the function U in (2.4) by $\lambda/2h(q)$, $M(\eta,p)$ by $(p^2+1)h(\eta)$ and $b(p)$ by unity.

3. Representation of the eigenfunctions for different values of parameters

First we consider the case $n \sim O(1)$. Equations (1.12) and (1.13) show that for such $n, q = O(\epsilon)$, $\lambda_0 = O(\epsilon)$. As Miles (1989) has shown,

$$\lambda_0 = \epsilon(2n+1) + \epsilon^2 h''(0)(n^2+n+\tfrac{1}{2}) + \dots, \quad q = \epsilon\kappa^{-2}(2n+1) + \dots.$$

As we have already noted in §1, equation (1.15) after rather lengthy but elementary calculations gives $\lambda_1 = \frac{1}{4}h''(0) + O(\epsilon)$, and (1.14) transforms into (1.9). Consider the formula for the eigenfunctions. We have $S/\epsilon = (2n+1)\tan^{-1} p + O(\epsilon)$, $a_0 = \kappa^{-1} + O(\epsilon)$. Dropping the $O(\epsilon)$ terms and taking into account the equality $\tan^{-1} p = (1/2\mathrm{i}) \ln (\mathrm{i}-p)/(\mathrm{i}+p)$, we get

$$Z(\eta) \sim \frac{1}{\mathrm{i}(2\pi\epsilon)^{\frac{1}{2}}} \int_C \frac{(\mathrm{i}+p)^n}{(\mathrm{i}-p)^{n+1}} \exp(\mathrm{i}p\eta/\epsilon)\,\mathrm{d}p.$$

This expression is equal to $(2\pi/\epsilon)^{\frac{1}{2}}$ multiplied by the residue of the integrand at $p = \mathrm{i}$. The residue is readily calculated and we get the formula (1.19) modulo a factor.

Consider now large $n \sim 1/\epsilon$. The stationary points $p_{1,2}$ of the integral representing Z are given by $\eta = q(p, \lambda_0)$. Obviously, $p_1 = -p_2$. These stationary points are non-degenerate for $\delta \leqslant \eta \leqslant q_0 - \delta$, where δ is an arbitrary positive number. Carrying through the appropriate standard calculation, for such η we get Keller's (1958) formula

$$Z \sim \frac{2}{[M_p(\eta, p_1(\eta))]^{\frac{1}{2}}} \cos\left[\frac{1}{\epsilon}\int_{q_0}^{\eta} p_1(\eta)\,\mathrm{d}\eta - \tfrac{1}{4}\pi\right].$$

For $\eta = q_0$ the stationary point $p_1 = p_2 = 0$ is degenerate, $S''(0) = 0$. Expanding the phase and the amplitude in Taylor series in p and leaving only the terms of order p^3 and 1 respectively, we get quite standardly for $|\eta - q_0| \ll 1$

$$Z \sim \frac{(2\pi)^{\frac{1}{2}} a_0(0)}{\epsilon^{\frac{1}{6}} \alpha^{\frac{1}{3}}} \operatorname{Ai}\left(\frac{\eta - q_0}{\epsilon^{\frac{2}{3}} \alpha^{\frac{1}{3}}}\right),$$

where $\alpha = -q_{pp}(0, \lambda_0) > 0$. In the small vicinity of the point $\eta = 0$ the stationary points are also degenerate. We note that only the integral along the arc gives a significant contribution to the asymptotics at small η. For large $|p|$ we can leave only the leading terms in the expansions of the phase and the amplitude: $a_0 \sim \pm 1/p$, $S \sim S(\pm\infty) - \lambda_0/p, p \rightarrow \infty$, $\operatorname{Re} p \gtrless 0$. Thus we get

$$Z \sim \frac{1}{(2\pi\epsilon)^{\frac{1}{2}}}(-1)^{n+1}\,\mathrm{i}\int \frac{1}{p}\exp\left(\frac{\mathrm{i}p\eta}{\epsilon} + \frac{\mathrm{i}\lambda_0}{\epsilon p}\right)\mathrm{d}p. \tag{3.1}$$

Here the integral is taken along the arc $|p| = R, \operatorname{Im} p > 0$. We note that the same integral taken along the arc $|p| = R, \operatorname{Im} p < 0$ is of order $\epsilon^{\frac{1}{2}}$ for $\eta < \lambda_0/R^2$. Therefore, changing Z by $O(\epsilon^{\frac{1}{2}})$, we can assume that the integration in (3.1) is carried along the circle $|p| = R$. Using the well-known integral representation of the Bessel function

$$J_0(z) = \frac{1}{2\pi\mathrm{i}}\int_{C_1} \frac{1}{p}\exp\left[\mathrm{i}z\left(p + \frac{1}{p}\right)\right]\mathrm{d}p,$$

where C_1 is the unit circle, we get finally for $0 \leqslant \eta \ll 1$

$$Z(\eta) \sim (2\pi\epsilon)^{\frac{1}{2}}(-1)^n J_0(2\lambda_0 \eta^{\frac{1}{2}}/\epsilon). \tag{3.2}$$

This fact is not at all surprising, the function (3.2) being the inner asymptotic for the shallow-water equation (1.8).

4. Edge waves in a rotating basin

The linearized equations of stratified or rotating fluid motions for certain values of the parameters involved can be reduced to the boundary-value problem for equation (1.1) with condition (1.2) at the free surface and a mixed condition at the bottom. Greenspan (1970) and Evans (1989) have constructed the analogues of Ursell's set of edge waves trapped by a beach of constant slope for exponentially stratified and rotating fluid, respectively. We consider here only the case of rotation (the case of exponential stratification can also be treated in a similar way).

Evans (1989)† has reduced the initial linearized system of equations describing the motion of a fluid in a basin rotating with angular velocity Ω directed vertically

† Note that there are two misprints in Evans' paper: in Appendix B $\epsilon = 2\Omega/\omega$ and $\alpha = \pm k\epsilon \sin\beta$ must be used instead of Ω/ω and $\pm k \sin\beta$, respectively.

upwards, to a system which in the coordinates $\eta = \epsilon ky$, $\zeta = \mu kz$, $\mu = (1-\gamma^2)^{-\frac{1}{2}}$, $\gamma = 2\Omega/\omega$ coincides with (1.1), (1.2) (the function h being replaced by μh and λ by $\nu = \lambda/\mu$). Instead of (1.3) the bottom condition has the form

$$\phi_\zeta \pm \epsilon\gamma h'\phi + \epsilon^2 \mu h' \phi_\eta = 0, \quad \zeta = -\mu h(\eta), \tag{4.1}$$

where the $\pm$ signs correspond to the factor $\cos(\omega t \mp kx)$ in the solution. Seeking the solution of this system in the form (1.4) with λ replaced by ν we get (1.6) where instead of $L_{0,1}$ the functions

$$L_0^{\mathrm{r}} = \nu \cosh \mu\kappa h - \kappa \sinh \mu\kappa h,$$

$$L_1^{\mathrm{r}} = h'(\mu p - \mathrm{i}\gamma\mu)(\cosh \mu\kappa h - \nu\kappa^{-1} \sinh \mu\kappa h)$$

are used. The same argument as in §2 gives the equations

$$\kappa \tanh \mu\kappa h(q) = \nu_0, \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}p} a_m (M_\eta^{\mathrm{r}})^{\frac{1}{2}} = \mp \mathrm{i}\gamma\kappa^{-2} a_m + F_m^{\mathrm{r}}, \tag{4.3}$$

where $M^{\mathrm{r}} = \kappa \tanh \mu\kappa h(\eta)$ and $F_m^{\mathrm{r}} = 0$ for $m = 0$ and is given by the same formula as in (2.12) for $m > 0$ with M replaced by M^{r}, b by $b_{\mathrm{r}} = \cosh \mu\kappa h$ and L_l by L_l^{r} according to

$$L_2^{\mathrm{r}} = -\nu_1 \cosh \mu\kappa h, \quad L_l^{\mathrm{r}} = \mathrm{i}^{-l}[\nu_{l-1} \cosh \mu\kappa h - \mathrm{i}\nu_{l-2}(p - \mathrm{i}\gamma)\mu h' \sinh \mu k h], \quad l \geqslant 3.$$

The solutions of (4.3) are

$$a_0 = (M_\eta^{\mathrm{r}})^{-\frac{1}{2}} \exp(\mp \mathrm{i}\gamma \tan^{-1} p),$$

$$a_m = (M_\eta^{\mathrm{r}})^{-\frac{1}{2}} \exp(\mp \mathrm{i}\gamma \tan^{-1} p) \int_0^p F_m^{\mathrm{r}} \exp(\pm \mathrm{i} \tan^{-1} p)\, \mathrm{d}p.$$

The conditions of the analyticity of the function Z give

$$\left.\begin{aligned} \frac{1}{\pi\epsilon}\int_{-\infty}^{+\infty} q(p, \nu_0)\, \mathrm{d}p &= 2n + 1 \pm \gamma, \\ \nu_m \int_{-\infty}^{+\infty} |a_0|^2\, \mathrm{d}p &= -\int_{-\infty}^{+\infty} G_m^{\mathrm{r}} a_0 \exp(\pm 2\mathrm{i}\gamma \tan^{-1} p)/b_{\mathrm{r}}\, \mathrm{d}p. \end{aligned}\right\} \tag{4.4}$$

Here G_m^{r} is the same as in (2.12) with appropriate replacements.

For small n we get $\nu = \epsilon\,(2n + 1 \pm \gamma) + O(\epsilon^2)$ and for

$$h = \eta, \ \nu = \sin \epsilon\mu(2n + 1 \pm \gamma) + O(\epsilon^2)$$

for all n. The last formula coincides with accuracy $O(\epsilon^2)$ with Evans' (1989) result applied to the case of small slope.

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